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Analysis of partial differential equation models in macroeconomics

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Abstract
Macroeconomics theory has focused on studying system of difference equations or ordinary differential equations describing the evolution of a relatively small number of macroeconomics aggregates. These system are typically derived from the optional control problem of a representing agent,; the purpose of a is to get mathematicians interested in studying a number of partial differential equations (PDEs) that naturally arise in macroeconomics these PDEs come from models designed to study some of the most important questions in Economics.

Keywords: Partial differential equations (PDEs), macroeconomics, studying system

Introductions
Macroeconomic is concerned with some of the most important questions in economics for example: what causes recessions and what should be done about them? Why are some countries so much poorer than others. Heterogeneous agent models are usually set in discrete time. While they are workhorses of modern macroeconomics, relatively little is known about their theoretical properties and they often prove difficult to compute. To make progress, some recent papers have therefore studied continuous time versions of such models. Our paper reviews this literature. Macroeconomic models with heterogeneous agents share a common mathematical structure which, in continuous time, can be summarized by a system of coupled nonlinear partial differential equations (PDEs): (i) a Hamilton–Jacobi–Bellman (HJB) equation describing the optimal control problem of a single atomistic individual and (ii) an equation describing the evolution of the distribution of a vector of individual state variables in the population (such as a Fokker–Planck equation, Fisher–KPP equation or Boltzmann equation). While plenty is known about the properties of each type of equation individually, our understanding of the coupled system is much more limited. Lasry & Lions [1-3] and Lions [8] have termed such a system a 'mean field game' and obtained some theoretical characterizations for special cases, but many open questions remain. For useful reference on mean field games, one can see for example Bardi, Gueant *et al*. Gomes *et al*. and Cardaliaguet. The purpose of this article is to present important examples of these systems of PDEs that arise naturally in macroeconomics, to discuss what is known about their properties, and to highlight some directions for future research.

Analysis
Macroeconomics is the study of large economic system. Most commonly, this system is the economy of a country. But, it many also be a more complex system such as the world as a whole. The discrete time model of Aiyagari, Bewley and Huggett is one of the workhorses of modern macroeconomics. This model captures in a relatively parsimonious way the evolution of the income and wealth distribution and its effect on macroeconomic aggregates. It is a natural framework to study the effect of various policies and institutions on inequality. A huge number of problems in macroeconomics have a similar structure and so this is a particularly useful starting point. The simplest formulation of the model is due to Huggett and we here present a continuous time formulation of Huggett's model presented in Achdou *et al*. There is a continuum of infinitely lived households that are heterogeneous in their wealth a and their income z. Households solve the following optimization problem:

$$\max_{\{c_t\}} \quad \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t)\, dt \quad \text{s.t.}$$
$$da_t = (z_t + r(t)a_t - c_t)\, dt$$
$$dz_t = \mu(z_t)\, dt + \sigma(z_t)\, dW_t$$
$$a_t \geq \underline{a}.$$

Households have utility functions u(c) over consumption c that are strictly increasing and strictly concave (e.g. u(c) =c1−γ /(1 − γ), γ >0) and they maximize the present discounted value of utility from consumption, discounted at rate ρ. Households can borrow and save at an interest rate r(t) which is determined in equilibrium and they optimally choose how to split their total income z_t + r(t)a_t between consumption and saving. Their income evolves exogenously according to a diffusion process dz_t =μ(z_t) dt + σ(z_t) dW_t in a closed interval [z,¯z] (it is reflected at the boundaries if it ever reaches them). Importantly, there is a state constraint a ≥ a for some scalar −∞<a ≤ 0. This state constraint has the economic interpretation of a borrowing constraint, e.g. if a=0 households can only save and cannot borrow at all. The interest rate r(t) must be such that the following equilibrium condition is satisfied:

$$\int ag(a,z,t)\, da\, dz = 0,$$

where *g(a, z, t)* denotes the cross-sectional distribution of households with wealth a and income z at time t. The interpretation of this equilibrium condition is as follows: wealth a here takes the form of bonds, and the equilibrium interest rate r is such that bonds are in zero net supply. That is, for every dollar borrowed, there is someone else who saves a dollar. The equilibrium can be characterized in terms of an HJB equation for the value function v and a Fokker–Planck equation for the density of households g. In a stationary equilibrium, the unknown functions v and g and the unknown scalar r satisfy the following system of coupled PDEs (stationary mean field game) on (a,∞) × (z, ¯z):

$$\frac{1}{2}\sigma^2(z)\partial_{zz}v + \mu(z)\partial_z v + (z+ra)\partial_a v + H(\partial_a v) - \rho v = 0, \tag{1}$$

$$-\frac{1}{2}\partial_{zz}(\sigma^2(z)g) + \partial_z(\mu(z)g) + \partial_a((z+ra)g) + \partial_a(\partial_p H(\partial_a v)g) = 0, \tag{2}$$

$$\int g(a,z)\, da\, dz = 1, \quad g \geq 0 \tag{3}$$

And $\int ag(a,z)\, da\, dz = 0,$ (4)

Where the Hamiltonian H is given by

$$H(p) = \max_{c\geq 0}(-pc + u(c)). \tag{5}$$

The function v satisfies a state constraint boundary condition at a=a and Neumann boundary conditions at z=z and z= ¯z. The function v satisfies a state constraint boundary condition at a=a and Neumann boundary conditions at z=z and z= ¯z. In general, the boundary value problem including the Bellman equation and the boundary condition has to be understood in the sense of viscosity. Whereas the boundary problem with the Fokker–Planck equation is set in the sense of distributions. An important issue is to check that actually yields an optimal control (verification theorem): this is a direct application of Itô's formula if v is smooth enough; for general viscosity solutions, one may apply the results of Bouchard & Touzi and Touzi (this has not been done yet). With well-chosen initial and terminal conditions, solutions to the HJB equation are expected to be smooth and we therefore look for such smooth solutions. If v is indeed smooth, the state constraint boundary condition can be shown to imply

$$(z+r\underline{a})\lambda + H(\lambda) \geq (w+r\underline{a})\partial_a v(\underline{a},z) + H(\partial_a v(\underline{a},z)) \quad \forall \lambda \geq \partial_a v(\underline{a},z)$$

or equivalently,

$$z + ra + \partial_p H(\partial_a v) \geq 0, \quad a = \underline{a} \tag{6}$$

so that the trajectory of a points towards the interior of the state space. Finally, note that the interest rate r—which is determined by the equilibrium condition is the only variable through which the distribution g enters the HJB equation. The time-dependent analogue of is also of interest. In the time-dependent equilibrium, the unknown functions v and g satisfy the following system of coupled PDEs (time-dependent mean field game) on (a,∞) × (z, ¯z) × (0, T):

$$\partial_t v + \frac{1}{2}\sigma^2(z)\partial_{zz}v + \mu(z)\partial_z v + (z+r(t)a)\partial_a v + H(\partial_a v) - \rho v = 0, \tag{7}$$

$$\partial_t g - \frac{1}{2}\partial_{zz}(\sigma^2(z)g) + \partial_z(\mu(z)g) + \partial_a((z+r(t)a)g) + \partial_a(\partial_p H(\partial_a v)g) = 0, \tag{8}$$

$$\int g(a,z,t)\, da\, dz = 1, \quad g \geq 0 \tag{9}$$

and $\int ag(a,z,t)\, da\, dz = 0,$ (10)

where the Hamiltonian H is given by. The density g satisfies the initial condition g(a, z, 0)= g0(a, z). For the terminal condition for the value function v, we generally take T large and impose *v(a, z, T)=v∞(a, z)*, where v_∞ is the stationary value function, i.e. the solution to the stationary problem (1)–(4). The function v also still satisfies the state constraint boundary condition (6) and Neumann boundary conditions at z=z and z= ¯z.

Conclusion

We have surveyed a large literature in macroeconomics that studies theories that explicitly model the equilibrium interaction of heterogeneous agents. These theories share a common mathematical structure which can be summarize by a system of coupled nonlinear PDEs or mean field game. Some of our examples are well-understood problems in the theory of PDEs, whereas others present new and challenging mathematical problems.

References

1. Lasry JM, Lions PL. Jeux à champ moyen. I. Le cas stationnaire. C. R. Math. Acad. Sci. Paris. 2006;(343):619-625. (doi:10.1016/j.crma.2006.09.019)

2. Lasry JM, Lions PL. Jeux à champ moyen. II. Horizon fini et controle optimal. C. R. Math. Acad. Sci. Paris. 2006;(343):679-684. (doi:10.1016/j.crma.2006.09.018)
3. Lasry JM, Lions PL. Mean field games. Jpn J Math. 2007;(2):229-260. (doi:10.1007/s11537-007-0657-8)
4. Lions PL. Cours du Collège de France. 2007-2011.See http://www.college-de-france. fr/default/EN/all/equ_der/.
5. Bardi M. Explicit solutions of some linear-quadratic mean field games. Netw. Heterog. Media. 2012;(7):243-261. (doi:10.3934/nhm.2012.7.243)
6. Y Skwame, PI Dalatu, J Sabo, M Mathew. Numerical application of third derivative hybrid block methods on third order initial value problem of ordinary differential equations. Int J Stat Appl Math 2019;4(6):90-100.
7. Gueant O, Lasry JM, Lions PL. Mean field games and applications. In Paris-Princeton Lectures on Mathematical Finance 2010. Lecture Notes in Mathematics. Berlin, Germany: Springer. 2003-2011, 205-266.
8. Gomes DA, Mohr J, Souza RR. Discrete time, finite state space mean field games. J. Math. Pures Appl. 2010;(93):308-328. (doi:10.1016/j.matpur.2009.10.010)
9. Cardaliaguet P. Notes on mean field games. Preprint. 2010.
10. Gueant O. A reference case formean field games models. J Math. Pures Appl. 2009;(92)276-294. (doi:10.1016/j.matpur.2009.04.008)