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Semi-classical lattice Boltzmann model for dissipative quantum plasma flow

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Abstract
A semi-classical lattice Boltzmann model for quantum plasma flow is proposed based on the Wigner-Fokker-Planck equation coupled with Poisson's equation. We derive the discrete quantum kinetic equation with operator terms series-expanded by Hermitian polynomials and present the semi-classical lattice Boltzmann equation that guarantees numerical stability. Numerical experiments are performed on electron fluid in dense quantum plasma, which are typically generated by a strong laser or present inside white dwarfs and supernovas. The different power spectra in electrostatic quantum fluid are analyzed through comparison with power laws. Our numerical modeling open a new avenue for studying the transport physics in semiconductors and the quantum optics as well as for astrophysical and laboratory-generated quantum plasmas.

Keywords: Quantum plasma, lattice Boltzmann method, Wigner formalism, semi-classical, numerical simulation

1. Introductions

The dynamics for quantum condensed fluids has been employed as a valuable tool to describe quantum transport in physics of semiconductor, nuclear, plasma and light [1-6]. Application of the standard quantum mechanics based on Hamiltonian theory to dissipative quantum fluids leads to considerable conceptual difficulties, whereas the Wigner formalism[7, 8], in comparison to those of Schrödinger or Heisenberg, is suitable for employing classical methodologies to quantum systems due to its similarity to the universal kinetic theory. Because the Wigner distribution function (WDF) can also take negative values, it has no obvious physical meaning as one of classical Boltzmann kinetic theory (see Ref. [9] for detailed physical analysis of the WDF). Nevertheless, it is hardly limited to calculate macroscopic quantities (e.g. density or current) of interest in various disciplines by means of the WDF [10-14]. In special, the Wigner formalism seems to be an effective mathematical tool in describing and treating advection-diffusion-reaction systems from the point of view of quantum mechanics. Recollecting the basic models such as Navier-Stokes equation derived from classic kinetic equation, the numerical modeling [15-19] based on the Wigner formalism give us a sense that can directly apply computational hydrodynamic methods to the quantum fluid system. The Wigner-Fokker-Planck equation (WFPE) [20-24] with scattering kernel added to the standard model has often been applied to quantum dissipative system.

From a classical point of view, the most successful mathematical model for dissipative fluid is probably the Navier-Stokes equation, meanwhile the most effective numerical model for solving the Navier-Stokes equation may be the lattice Boltzmann scheme [25]. Its mathematical model has been constantly updated [26, 27] and applied extensively to different fields [28-30]. The so-called quantum lattice Boltzmann scheme has been used as an approximation for solving the Schrödinger equation with a form similar to the Euler equation for classical fluid [31-33]. However, the formal property of distribution function corresponding to the wave function seems to devalue its significance. Although the lattice Boltzmann method has started from lattice gas automation [25], it is clearly a special discrete form of the Boltzmann kinetic equation. This fact has provided the possibility to apply the lattice Boltzmann model to quantum kinetic theory [40, 41].

In this paper, we develop a semi-classical lattice Boltzmann model (SCLBM) that is

available for analyzing dissipative electrostatic quantum fluids based on the Wigner formalism. Applying the Hermite-series expansion to the operator terms within the Wigner-Fokker-Planck equation (WFPE) coupled with the Poisson's equation, a discrete WFPE is obtained and its equivalence to the quantum fluid dynamics equation system is verified through the Chapman-Enskog analysis. An alternative SCLBM is found by a few algebraic manipulations to provide moment conservation and numerical stability. We perform our numerical experiment with dense plasma evolved from the initial Taylor-Green vortex flow and analyze the power and its spectra associated with typical energies in electrostatic quantum fluid systems.

2. Mathematical Modeling

We start from the quantum kinetic equation based on the Wigner formalism [7]. For a nonrelativistic electron ensemble undergoing potential energy $U(\mathbf{r},t) = e\varphi(\mathbf{r},t)$, where e the elementary charge and φ the electrostatic potential, and interacting with a thermal bath of oscillator, the WDF $f_w = f_w(\mathbf{r}, \mathbf{v}, t)$ is evolved by WFPE [20, 34- 36]:

$$\partial_t f_w + \mathbf{v} \cdot \nabla_{\mathbf{r}} f_w = -\Theta[U] f_w + \mathcal{L}_{QFP}[f_w], \tag{1}$$

where, $\mathbf{r}$ and $\mathbf{v}$ the coordinates and velocity vectors, respectively; ∂_t, ∇_r and ∇_v the partial differential operators with respect to time, coordinates and velocity, respectively. The first term on right-hand side in Eq. (1) is given by

$$\Theta[U] = \frac{i}{\hbar}\left[U(\mathbf{r} + \frac{i\hbar}{2}\nabla_v, t) - U(\mathbf{r} - \frac{i\hbar}{2}\nabla_v, t)\right], \tag{2}$$

which reflects the uncertainty relation between position and velocity[8]. Expanding two terms in the operator $\Theta[U]$ into the Taylor series, one reads easily:

$$\Theta[U] = \sum_{k=0}^{\infty} A_k \hbar^{2k} m^{2k+1} \nabla_r^{2k+1} U(\mathbf{r},t) \cdot \nabla_v^{2k+1}, \tag{3}$$

where, m the effective mass of electron; $\hbar$ the Planck constant divided by 2π; $A_k = (-1)^{k+1}/2^{2k}(2k+1)!$. The quantum Focker-Planck operator in Eq.(1) is expressed as [20]:

$$\mathcal{L}_{QFP} = D_{vv}\nabla_v^2 + \frac{\eta}{m}\mathbf{v} \cdot \nabla_v + D_{rv}\nabla_r\nabla_v + D_{rr}\nabla_r^2, \tag{4}$$

with $D_{vv} = \eta k_B T$, $D_{rv} = \frac{\eta\Omega\hbar^2}{6\pi m k_B T}$ and $D_{rr} = \frac{\eta\hbar^2}{12m^2 k_B T}$ where, η the coupling(dumping) rate of the bath; k_B the Boltzmann constant; T the temperature; Ω the cut-off frequency of recervior oscillator. In the same way as the classical formulation from the kinetics model to the hydrodynamics, the number (probability) density, *n*, and the flux (or current), **J**, are connected with the distribution function f_w:

$$n(\mathbf{r},t) = \int f_w(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}, \tag{5}$$

$$\mathbf{J}(\mathbf{r},t) = n(\mathbf{r},t)\mathbf{u}(\mathbf{r},t) = \int \mathbf{v} f_w(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}, \tag{6}$$

where $\mathbf{u}$ demonstrates the fluid flow velocity. The standard picture in classic statistics is that the system converges to the Maxwell-Boltzmann equilibrium while it is not for the Wigner's in strict sense. Moreover, the perfect positive definiteness of the Maxwellian distribution contrasts with the local negativity of the Wigner distribution [7] defined by

$$f_w(x,v,t) = \frac{m}{2\pi\hbar}\int ds\, exp\left(\frac{imvs}{\hbar}\right)\psi^*\left(x + \frac{s}{2}, t\right)\psi\left(x - \frac{s}{2}, t\right) \tag{7}$$

where, ψ the wave function; the star-script denotes conjugation. However, for ground or quantum statistical mixture states, one may take Maxwell-like distribution [37]:

$$f_w^{eq} \sim exp\left[-\frac{m}{\hbar\omega}tanh\left(\frac{\hbar\omega}{2k_B T}\right)(\mathbf{v} - \mathbf{u})^2\right] \tag{8}$$

Considering the prefactor in the exponent, $\xi_s = [2m\tanh(\hbar\omega/2k_B T)/\hbar\omega]^{-1/2}$ has a clear pattern of velocity and is reduced to the classical sound velocity $k_B T/m$ when $\hbar \to 0$. From this point of view, we define ξ_s as an effective sound

speed in the quantum fluid system. If the external potential acts on the electron ensemble by a homogeneous ion background with the mean number density n_0 and the charge e, in order to close the WFPE (1) on electrostatic quantum fluid, a Poisson equation is employed as:

$$\nabla_r^2\varphi = \frac{e}{\epsilon_0}[n(\mathbf{r},t) - n_0] \tag{9}$$

Here, ϵ_0 indicates the vacuum permittivity.

For non-dimensionalisation, we introduce scaled quantities of time t, velocity $\mathbf{v}$, coordinates $\mathbf{r}$, number density n, damping rate η, cut-off frequency Ω and Planck constant $\hbar$, respectively: $\tilde{t} = \omega_p t$, $\tilde{\mathbf{v}} = c_s\mathbf{v}/\xi_s$, $\tilde{\mathbf{r}} = c_s\omega_p\mathbf{r}/\xi_s$, $\tilde{\varphi} = ec_s^2\varphi/m\xi_s^2$, $\tilde{n} = n/n_0$, $\tilde{\eta} = \eta/m\omega_p$, $\tilde{\Omega} = \Omega/\omega_p$ and $\mathcal{H} = \omega_p c_s^2\hbar/m\xi_s^2$. Here, c_s the scaled effective sound speed; $\omega_p = (n_0e^2/m\varepsilon_0)^{1/2}$ the plasma frequency. The interaction constants, D_{vv}, D_{rv} and D_{rr}, are scaled as:

$$\widetilde{D}_{vv} = \Gamma^2 c_s^2\tilde{\eta},\ \widetilde{D}_{rv} = \frac{\tilde{\eta}\tilde{\Omega}\tilde{\hbar}^2}{6\pi\Gamma^2 c_s^2},\ \widetilde{D}_{rr} = \frac{\tilde{\eta}\tilde{\hbar}^2}{12\Gamma^2 c_s^2} \tag{10}$$

where, $\Gamma = \left[\frac{2k_BT}{\hbar\omega_p}\tanh\left(\frac{\hbar\omega_p}{2k_BT}\right)\right]^{1/2}$. Expanding Eq. (3) to second term ($k = 1$), one reads the scaled WFPE:

$$\partial_{\tilde{t}}f_w + \tilde{v}_\alpha\partial_{\tilde{x}_\alpha}f_w =$$

$$-\frac{\mathcal{H}^2}{24}\Phi_{\alpha\beta\gamma}\partial_{\tilde{v}_\alpha}\partial_{\tilde{v}_\beta}\partial_{\tilde{v}_\gamma}f_w + \widetilde{D}_{vv}\partial_{\tilde{v}_\alpha}^2 f_w + \Phi_\alpha\partial_{\tilde{v}_\alpha}f_w + \tilde{\eta}\partial_{\tilde{v}_\alpha}(\tilde{v}_\alpha f_w) + \widetilde{D}_{rv}\partial_{\tilde{x}_\alpha}\partial_{\tilde{v}_\alpha}f_w + \widetilde{D}_{rr}\partial_{\tilde{x}_\alpha}^2 f_w \tag{11}$$

where Φ_α and $\Phi_{\alpha\beta\gamma}$ indicate coordinates components of the vector $\nabla_r\tilde{\phi}$ and the tensor $\nabla_r^{(3)}\tilde{\varphi}$, respectively. To linearize the velocity derivative factors on the right side of Eq. (11), we introduce the Hermite series and l^{th}-order derivative of the WDF expressed by

$$f_w(\tilde{\mathbf{r}},\tilde{\mathbf{v}},\tilde{t}) = \omega\left(\frac{\tilde{\mathbf{v}}}{c_s}\right)\sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{a}_{(k)}(\tilde{\mathbf{r}},\tilde{t})\mathbf{H}_{(k)}\left(\frac{\tilde{\mathbf{v}}}{c_s}\right), \tag{12}$$

$$\nabla_{\tilde{v}}^{(l)}f_w = \omega\left(\frac{\tilde{\mathbf{v}}}{c_s}\right)\sum_{k=0}^{\infty}\frac{(-1)^l}{k!}\mathbf{a}_{(k)}(\tilde{\mathbf{r}},\tilde{t})\mathbf{H}_{(k+l)}\left(\frac{\tilde{\mathbf{v}}}{c_s}\right), \tag{13}$$

with the weight factor $\omega\left(\frac{\tilde{\mathbf{v}}}{c_s}\right) = (2\pi c_s^2)^{-d/2}\exp\left(-\frac{\mathbf{v}^2}{2c_s^2}\right)$, the Hermite polynomial $\mathbf{H}_{(k)}(\mathbf{x}) = (-1)^k\frac{1}{\omega(\mathbf{x})}\nabla^{(k)}\omega(\mathbf{x})$ and the expanding coefficient $\mathbf{a}_{(k)}(\tilde{\mathbf{r}},\tilde{t}) = \int f_w(\tilde{\mathbf{r}},\tilde{\mathbf{v}},\tilde{t})\mathbf{H}_{(k)}\left(\frac{\tilde{\mathbf{v}}}{c_s}\right)\mathrm{d}^d\left(\frac{\tilde{\mathbf{v}}}{c_s}\right)$ in *d*-dimension, respectively. Eq.(11) allows the definition for tensor contraction. For convenience, hereafter, we remove the swung dash for the coordinates $\mathbf{r}$ and time t, and the differential operator ∇only means for the coordinates, i.e. $\nabla \equiv \nabla_r$. After preceding the calculation of Hermitian polynomials, we discretized the WFPE (11) with respect to a set of the discrete velocity $\{\mathbf{c}_i\}$ corresponding $\mathbf{v}$, similar to the standard lattice Boltzmann scheme derived from the classical kinetic equation, as follows:

$$f_{wi}(\mathbf{r} + \mathbf{c}_i\Delta t, t + \Delta t) - f_{wi}(\mathbf{r},t) = \Delta t F_i \tag{14}$$

where, $F_i = \sum_{k=1}^{6}F_{ki}$ with,

$$F_{1i} = \frac{\omega_i\mathcal{H}^2\Phi_{\alpha\beta\gamma}\tilde{n}}{24c_s^6}\{c_s^2[c_{i\alpha}c_{i\beta}c_{i\gamma} - c_s^2(\delta_{\alpha\beta}c_{i\gamma} + \delta_{\beta\gamma}c_{i\alpha} + \delta_{\gamma\alpha}c_{i\beta})] + [c_{i\alpha}c_{i\beta}c_{i\gamma}c_{i\vartheta} - c_s^2(\delta_{\alpha\beta}c_{i\gamma}c_{i\vartheta} +$$
$$\delta_{\beta\gamma}c_{i\vartheta}c_{i\alpha} + \delta_{\gamma\vartheta}c_{i\alpha}c_{i\beta} + \delta_{\vartheta\alpha}c_{i\beta}c_{i\gamma} + \delta_{\alpha\gamma}c_{i\beta}c_{i\vartheta} + \delta_{\vartheta\beta}c_{i\gamma}c_{i\alpha})]\tilde{u}_\vartheta + c_s^4(\delta_{\alpha\beta}\delta_{\gamma\vartheta} + \delta_{\gamma\alpha}\delta_{\beta\vartheta} + \delta_{\vartheta\alpha}\delta_{\gamma\beta})\tilde{u}_\vartheta\}, \tag{15}$$

$$F_{2i} = \frac{\omega_i \tilde{D}_{vv} \tilde{n}}{c_s^5}\{c_s^2(c_{i\alpha}c_{i\beta} - c_s^2\delta_{\alpha\beta}) + [c_{i\alpha}c_{i\beta}c_{i\gamma} - c_s^2(c_{i\alpha}\delta_{\beta\gamma} + c_{i\beta}\delta_{\alpha\gamma} + c_{i\gamma}\delta_{\alpha\beta})]\tilde{u}_\gamma\}, \quad (16)$$

$$F_{3i} = -\frac{\omega_i \Phi_\alpha \tilde{n}}{c_s^4}[c_s^2 c_{i\alpha} + (c_{i\alpha}c_{i\beta} - c_s^2\delta_{\alpha\beta})\tilde{u}_\beta], \quad (17)$$

$$F_{2i} = \frac{\omega_i \tilde{D}_{vv} \tilde{n}}{c_s^5}\{c_s^2(c_{i\alpha}c_{i\beta} - c_s^2\delta_{\alpha\beta}) + [c_{i\alpha}c_{i\beta}c_{i\gamma} - c_s^2(c_{i\alpha}\delta_{\beta\gamma} + c_{i\beta}\delta_{\alpha\gamma} + c_{i\gamma}\delta_{\alpha\beta})]\tilde{u}_\gamma\}, \quad (18)$$

$$F_{5i} = -\omega_i \tilde{D}_{rv}\partial_\alpha \tilde{n}[c_s^2 c_{i\alpha} + (c_{i\alpha}c_{i\beta} - c_s^2\delta_{\alpha\beta})\tilde{u}_\beta], \quad (19)$$

$$F_{6i} = \tilde{D}_{rr}\partial_\alpha^2 f_{wi}, \quad (20)$$

where, Latin subscripts represent the coordinate components. In order to verify equivalence of the discrete WFPE (14) to the quantum hydrodynamics, and to obtain a lattice Boltzmann equation, we conduct Chapman-Enskog analysis employing the multi-scale expansions:

$$f_{wi} = f_{wi}^{(0)} + \varepsilon f_{wi}^{(0)} + \varepsilon^2 f_{wi}^{(0)} + \cdots,$$

$$\partial_t = \varepsilon\partial_{t1} + \varepsilon^2\partial_{t2},$$

$$\partial_\alpha = \varepsilon\partial_{\alpha 1}$$

$$F_i = \varepsilon F_i^{(1)} \quad (21)$$

where, ε is a small parameter.
The 2nd-order Taylor-expansion of Eq. (11) reads

$$D_i f_{wi} + \frac{\Delta t}{2} D_i^2 f_{wi} = F_i + O(\Delta t^2), \quad (22)$$

where, $D_i = \partial_t + c_{i\alpha}\partial_\alpha$. Applying the multi-scale expansion into Eq. (22), and straightening the order terms of ε, one reads that

$$\varepsilon: D_{i1} f_{wi}^{(0)} = F_i, \quad (23)$$

$$\varepsilon^2: \partial_{t_2} f_{wi}^{(0)} + D_{i1} f_{wi}^{(1)} + \frac{\Delta t}{2} D_{i1} F_i = 0, \quad (24)$$

where, $D_{i1} = \partial_{t_1} + c_{i\alpha}\partial_{\alpha 1}$. The moments of force are expressed by,

$$\sum_i F_i = \tilde{D}_{rr}\partial_\alpha^2 \tilde{n}, \quad (25)$$

$$\sum_i c_{i\alpha} F_i = -\tilde{n}\Phi_\alpha - \tilde{\eta}\tilde{n}\tilde{u}_\alpha - \tilde{D}_{rv}\partial_\alpha \tilde{n} + \tilde{D}_{rr}\partial_\alpha^2(\tilde{n}\tilde{u}_\alpha), \quad (26)$$

$$\sum_i c_{i\alpha} c_{i\beta} F_i = \frac{\Gamma^2\tilde{\eta}\tilde{n}}{c_s} - \tilde{n}\left(\tilde{u}_\alpha\Phi_\beta^{(1)} + \tilde{u}_\beta\Phi_\alpha^{(1)}\right) - \frac{\tilde{\eta}\mathcal{H}^2}{6\pi\Gamma^2}(\partial_\alpha\tilde{n}\tilde{u}_\beta + \partial_\beta\tilde{n}\tilde{u}_\alpha) + \frac{\tilde{\eta}\Omega\mathcal{H}^2}{12\Gamma^2}\partial_\gamma^2(\tilde{n}\tilde{u}_\alpha\tilde{u}_\beta + \tilde{n}c_s^2\delta_{\alpha\beta}), \quad (27)$$

Combinating Eqs. (22-27), equations of continuity and momentum for quantum fluid dynamics are obtained by:

$$\frac{\partial\tilde{n}}{\partial t} + \nabla(\tilde{n}\tilde{\mathbf{u}}) = \tilde{D}_{rr}\nabla^2\tilde{n}, \quad (28)$$

$$\frac{\partial(\tilde{n}\tilde{\mathbf{u}})}{\partial t}+\nabla(\tilde{n}\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}})=-\nabla\tilde{p}-\tilde{n}\nabla\tilde{\phi}-\tilde{\eta}\tilde{n}\tilde{\mathbf{u}}-\widetilde{D}_{rv}\nabla\tilde{n}+\widetilde{D}_{rr}\nabla^2(\tilde{n}\tilde{\mathbf{u}}), \quad (29)$$

Non-conservation of the continuity equation (28) creates some annoyance in applying the lattice Boltzmann scheme. Introducing $\tilde{\mathbf{u}}^*=\tilde{\mathbf{u}}-\widetilde{D}_{rr}\frac{\nabla\tilde{n}}{\tilde{n}}$ and $\nabla(\tilde{n}\tilde{\mathbf{u}})=\nabla(\tilde{n}\tilde{\mathbf{u}}^*)+\widetilde{D}_{rr}\nabla^2\tilde{n}$ to Eqs.(28, 29), and implementing some simple algebras, one reads that,

$$\frac{\partial\tilde{n}}{\partial t}+\nabla(\tilde{n}\tilde{\mathbf{u}}^*)=0, \quad (30)$$

$$\frac{\partial(\tilde{\mathbf{u}}^*)}{\partial t}+(\tilde{\mathbf{u}}^*\cdot\nabla)\tilde{\mathbf{u}}^*=-\nabla\tilde{p}-\nabla\varphi-\eta\tilde{\mathbf{u}}^*-D_{rv}\frac{\nabla\tilde{n}}{\tilde{n}}-\widetilde{D}_{rr}\tilde{\eta}\nabla\tilde{n}+\widetilde{D}_{rr}\nabla(\nabla\tilde{\mathbf{u}}^*+\nabla\tilde{\mathbf{u}}^{*\mathrm{T}})+\widetilde{D}_{rr}^2\frac{\nabla(\nabla^2\tilde{n})}{\tilde{n}}, \quad (31)$$

The sixth term on right wide in Eq.(31) operates as viscous one in the classical Navier-Stokes equation, so it possibility to apply the lattice Boltzmann scheme into Eqs.(30, 31).Therefore, with a new discrete distribution function f_i, we write a semi-classical lattice Boltzmann equation:

$$f_i(\mathbf{r}+\mathbf{c}_i\Delta t+t+\Delta t)-f_i(\mathbf{r},t)=$$

$$-\frac{\Delta t}{\tau}[f_i(\mathbf{r},t)-f_i^{eq}(\tilde{n},\tilde{\mathbf{u}}^*)]+(1-\frac{\Delta t}{2\tau})\mathcal{F}_i, \quad (32)$$

with $\mathcal{F}_i=\sum_{k=1}^{6}F_{ki}-\widetilde{D}_{rr}\partial_\alpha^2 f_i^{eq};\tau=\frac{\widetilde{D}_{rr}}{c_s^2}+\frac{\Delta t}{2}$. The equilibrium distribution function takes the same form with the standard lattice Boltzmann equation.

$$f_i^{eq}(\tilde{n},\tilde{\mathbf{u}}^*)=\omega_i\tilde{n}[1+\frac{\mathbf{c}_i\cdot\tilde{\mathbf{u}}^*}{c_s^2}+\frac{(\mathbf{c}_i\cdot\tilde{\mathbf{u}}^*)^2}{2c_s^4}-\frac{\tilde{u}^{*2}}{2c_s^2}], \quad (33)$$

where, $\tilde{n}=\sum_i f_i$ and $\tilde{\mathbf{u}}^*=\sum_i\mathbf{c}_i(f_i+\frac{\Delta t}{2}\mathcal{F}_i)$.

3. Numerical Experiment and Discussion

Our numerical experiment employing the SCLBM is conducted on a D2Q9 lattice with discrete velocity components $c_{ix}=\{0,1,0,-1,0,1,-1,-1,1\}$, and $c_{iy}=\{0,0,1,0,-1,1,1,-1,-1\}$, weight factor $\omega_i=\left\{\frac{4}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{36},\frac{1}{36},\frac{1}{36},\frac{1}{36}\right\}$, and sound speed $c_s=1/\sqrt{3}$.

We investigate two-dimensional electron fluid in dense plasma where the transverse degrees of freedom are neglected. The electron fluid flow is evolved from the Taylor-Green vortex, which is a benchmark flow where a simple initial condition develops in time to turbulent flow [38, 39]. In two-dimensional periodic quadrature geometry, it is initialized by the velocity field with components $\tilde{u}_{x0}^*=-\tilde{u}_0^*\cos x\sin y$ and $u_{y0}^*=\tilde{u}_0^*\sin x\cos y$, where $x,y\in[0,2\pi]$. We consider the dense plasma with mean electron number densities of $10^{30}\mathrm{m}^{-3}$, which are typical of intense laser-induced plasmas [42] and inside white dwarf and supernova [43]. The plasma is assumed to be maintained at the Fermi electron temperature $T_F\equiv\frac{\hbar^2}{2mk_B}(3\pi^2n_0)^{2/3}=423700\mathrm{K}$. The effective Planck constant and coupling constant are set to $H=0.3$ and $\Gamma=1.04$, respectively. The cut-off frequency Ω is much higher than the characteristic frequency of the electron, and is not larger than the thermal frequency $\tau_{\mathrm{th}}=\frac{k_BT_F}{\hbar}$ which sets the typical time scale for quantum effects. Thus, we take the dimensionless simulation parameters as $\widetilde{\Omega}=0.16$, $\tilde{\eta}=0.016$, $\widetilde{D}_{vv}=0.035$, $\widetilde{D}_{rv}=6.37\times10^{-4}$ and $\widetilde{D}_{rr}=0.002$.

The force acting on a dissipative electrostatic quantum fluid system is represented by the right-hand side of the momentum equation (31), and the spatial and temporal construction of this forcing can change scaling behavior of energy spectra in the inertial range. On the other hand, it can clearly violate the inertial range assumption due to its complete global nature, and hence the analysis for energy spectra in electrostatic quantum fluid is very interesting. We estimate the system by dividing it into the kinetic energy $E_k\sim\tilde{\mathbf{u}}^2$, the interaction energy $E_i\sim\tilde{n}^2$, the electrical energy $E_e\sim(\nabla\tilde{\varphi})^2$, and the quantum energy

$E_q \sim (\nabla \tilde{n})^2$. Fig.1 shows that kinetic and interaction energies are cascaded to a larger scale compared to electrical and quantum energies. With the two-dimensional system in mind, quantum fluid turbulence also coexists with small and large scale structures, as well as classical one. Generally, the cascade occurs in forced turbulence simulations where certain modes are excited through external forces in the spectral space. Randomly-excited Fourier modes transfer spectral energy by keeping the kinetic constant in the wavenumber space, while the energy contained in large-scale vortices is transferred to a small scale, resulting in a statistically steady inertial regime. To quantify the electrostatic quantum fluid turbulence, we consider power spectra associated with the kinetic, interaction, electrical and quantum energies of the electron fluid turbulence in dense plasma. The energy spectral densities are defined by $E(k) = \sum_{k' \le k \le k'+1} E(\mathbf{r}) \exp(i\mathbf{k}^{'} \cdot \mathbf{r})$, where $k = |\mathbf{k}| = \sqrt{k_x^2 + k_y^2}$.

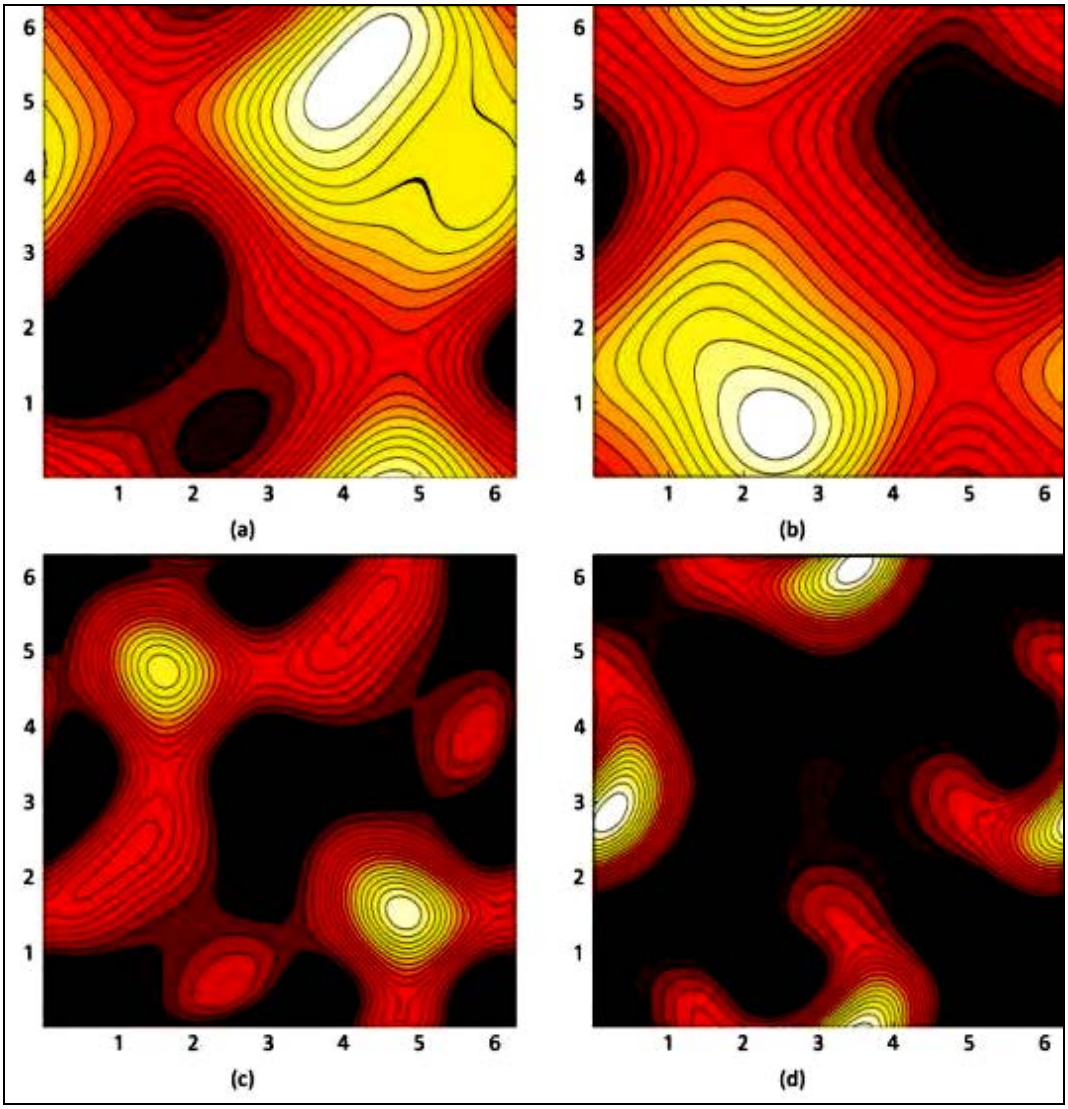

Fig 1: (Color online) Energy contours of electron fluid in two-dimensional dense plasma of mean electron number density $10^{30}m^{-3}$ at Fermi temperature. (a) Kinetic energy, $E_k \sim \tilde{\mathbf{u}}^2$; (b) Interaction energy, $E_i \sim \tilde{n}^2$; (c) Electrical energy, $E_e \sim (\nabla \tilde{\varphi})^2$; (d) Quantum energy, $E_q \sim (\nabla \tilde{n})^2$. The kinetic and interaction energies are cascaded to a larger scale compared to the electrical and quantum energies.

As shown in Fig.2, cascades with scaling behavior k^{-3} occurs both for kinetic and interaction energies, which is similar to that in classical fluid turbulence [44] where the two-dimensional Navier-Stokes framework is adopted. From a classical point of view, the simultaneous conservation of energy and enstrophy by the advection term of the two-dimensional Navier-Stokes equation results in a double cascade of kinetic energy under external forcing. In other words, it causes the inverse cascade of kinetic energy to a large-scale and the direct cascade of enstrophy to a small-scale. For electrical energy ($k^{-3.5}$), the deviation from the scaling k^{-3} is thought to be related to strong, long-lived vortices present in its initial spectrum. The spectral slope of the quantum energy is close to the universal Kolmogorov power law ($k^{-5.3}$).

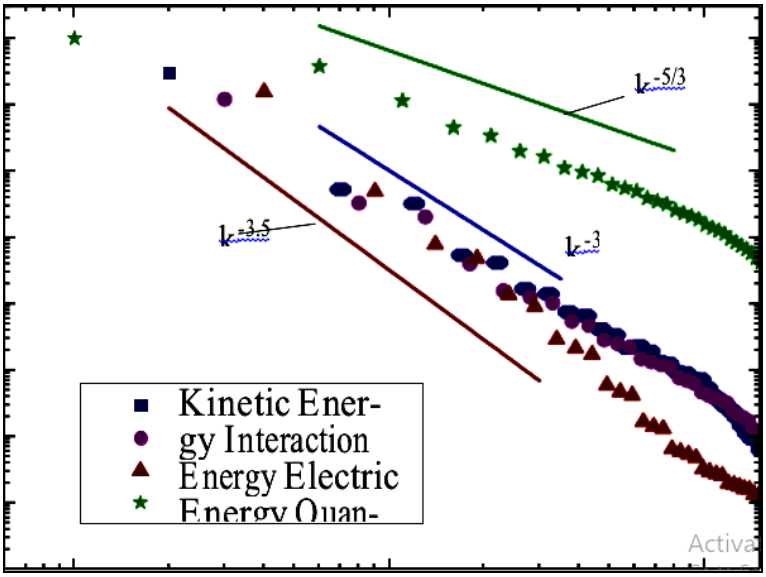

Fig 2: Power spectra associated with different energies of quantum electron fluid in dense plasma: Kinetic energy, $E_k \sim \tilde{\mathbf{u}}^2$; Interaction energy, $E_i \sim \tilde{n}^2$; Electrical energy, $E_e \sim (\nabla\tilde{\varphi})^2$; Quantum energy, $E_q \sim (\nabla\tilde{n})^2$.

Both the kinetic and interaction energies are cascaded by a scaling k^{-3}, and the electrical and quantum energies follow the Kraichnan power law($k^{-3.5}$) and the Kolmogorov power law ($k^{-5/3}$), respectively.

4. Summary and Conclusion

As mentioned by some researchers, the lattice Boltzmann method may be the most effective numerical approximation for the Navier-Stokes equation. Our key contribution beyond the traditional lattice Boltzmann model is the effective coupling of the quantum fluid system and the classical numerical scheme. The series expansion and velocity-space discretization of operator terms in the quantum kinetic equation and the employment of Maxwell-Boltzmann-like equilibrium distribution functions for quantum harmonic oscillators have enabled the introduction of the lattice Boltzmann scheme, a powerful solver of the classical Navier-Stokes equation, into the quantum fluid system. Furthermore, the direct numerical simulation for electrostatic fluid based on our SCLBM may be a contribution to quantum plasma physics. Analysis of the different power spectra characterizing turbulence will be an interesting topic for advance in the kinetics and dynamics of quantum plasmas. Further studies including three-dimensional geometry and other particle species (ion, neutron, elementary particle, etc.) are required. On other hand, we will also update our model to analyze the kinetic and magneto-hydrodynamic behaviors of quantum plasmas in future work.

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Author Contribution

Wi-Song Choe: Data curation; Formal analysis; Investigation; Software; Visualization; Writing-original draft. Suk Kim: Formal analysis; Investigation; Methodology. Yu-Bong Kim: Methodology; Formal analysis; Software. Yong-Jun Kim: Conceptualization; Data curation; Validation; Writing-review & editing; Ji-Hyok Kim: Software; Validation.

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Declaration of competing interest

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Data Availability Statement

Data will be made available on request.

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