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Geometric aspects of warped product semi-slant Submanifolds of Kenmotsu manifolds

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Abstract
In this manuscript, we have executed an extensive revision of [1] and carried out a detailed examination of the attributes of warped product semi-slant submanifolds within the paradigm of Kenmotsu manifolds. We have obtained notable insights regarding the presence of warped product semi-slant submanifolds within Kenmotsu manifolds, expressed in relation to the fundamental canonical structure.
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1. Introductions
The differential geometry pertaining to slant submanifolds has demonstrated sig-nificant progress since B. Y. Chen delineated slant immersion within the realm of complex geometry as a coherent generalization of holomorphic immersions and totally real immersions [2, 3, 4]. Bishop and O'Neill [5] pioneered the concept of warped product manifolds, while the notion of semi-slant submanifolds was acknowledged in 1994 through the research conducted by N. Papaghuic [6]. Specifically, semi-slant submanifolds in almost Hermitian manifolds are defined based on the principles of CR-submanifolds [7]. In 1996, A. Lotta elucidated the concept of slant immersion of a Riemannian manifold into an almost contact metric manifold [8]. In the year 1999, J. L. Cabrerizo *et al*. investigated semi-slant submanifolds through the lens of almost contact metric manifolds [3]. B. Sahin brought to light the examinations concerning warped product semi-slant submanifolds within Kaehler manifolds [9]. Subsequently, K. A. Khan *et al*. scru-tinized warped product semi-slant submanifolds in the context of cosymplectic
manifolds and articulated the non-existence of proper warped product semi-slant submanifolds of the forms $N_T \times_f N_\theta$ and $N_\theta \times_f N_T$ within cosymplectic manifolds (see) [10].
Recently, M. Atceken established that the warped product submanifolds of the types $M = N_T \times_f N_\theta$ and $M = N_\theta \times_f N_\perp$ associated with a Kenmotsu manifold $\tilde{M}$ do not exist, where the manifolds N_θ and N_T (resp. $N_\perp$) are clas-sified as proper slant and φ-invariant (resp. anti-invariant) submanifolds of a
Kenmotsu manifold $\tilde{M}$, respectively (see) [11]. In the present study, we have achieved notable findings regarding the existence of warped product semi-slant submanifolds within the framework of Kenmotsu manifolds.

2. Preliminaries
Let $\tilde{M}$ be a $(2n + 1)$-dimensional C^∞ manifold that possesses an almost contact metric structure denoted as (ϕ, ξ, η, g), where ϕ represents a tensor field of type (1, 1), ξ signifies a vector field, η corresponds to a 1-form, and g constitutes a
Riemannian metric on $\tilde{M}$; all of these tensor fields conform to the conditions specified in [12]

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \eta o \varphi = 0, \varphi\xi = 0, \tag{2.1}$$

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \varphi Y) = -g(\varphi X, Y), g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in T\widetilde{M}$. Here, $T\widetilde{M}$ denotes the tangent bundle associated with the manifold $\widetilde{M}$. The notation Φ refers to the fundamental two forms, which is expressed by the relation $g(X, \phi Y) = \Phi(X, Y)$. The manifold is classified as contact if the condition $\Phi = d\eta$ is satisfied. Should ξ be a Killing vector field with respect to the metric g, the corresponding contact metric structure is termed a *K*-contact structure. It is established that a contact metric manifold qualifies as *K*-contact if and only if the relation $\widetilde{\nabla}_X \xi = -\phi X$ holds true, where $\widetilde{\nabla}$ indicates the Levi-Civita connection on $\widetilde{M}$. An almost contact structure on $\widetilde{M}$ is characterized as normal if the equation $[\phi, \phi] + 2d\eta \otimes \xi = 0$ is fulfilled, where $[\phi, \phi]$ represents the Nijenhuis tensor corresponding to ϕ. A Sasakian manifold is defined as a normal almost contact metric manifold. It is noteworthy that every Sasakian manifold inherently possesses the property of being *K*-contact. Furthermore, a three-dimensional *K*-contact manifold is identified as Sasakian. An almost contact metric manifold is designated as Sasakian if and only if the following condition is satisfied:

$$\widetilde{\nabla}_X \varphi Y = g(X, Y)\xi - \eta(Y)X; X, Y \in T\widetilde{M}. \quad (2.4)$$

Moreover, within the context of a Sasakian manifold, the following relation holds:

$$\widetilde{\nabla}_X \xi = -\varphi X. \quad (2.5)$$

For any vector $X \in T\widetilde{M}$, ξ serves as the structure vector field. An almost contact metric structure (ϕ, ξ, η, g) on $\widetilde{M}$ is referred to as a β-Kenmotsu manifold [13] if the following condition is satisfied:

$$(\bar{\nabla}_X \varphi)Y = \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}, \quad (2.6)$$

where β is a smooth function defined on $\widetilde{M}$ and $\bar{\nabla}$ denotes covariant differenti-ation with respect to the metric g. Should one set $\beta = 1$ in (2.6), it follows that $\widetilde{M}$ is categorized as a Kenmotsu manifold. Let M be a submanifold immersed within a $(2n+1)$-dimensional contact metric manifold $\widetilde{M}$; we denote the induced metric on M by the same symbol g. TM represents the tangent bundle of the manifold M, while $T^{\perp}M$ constitutes the collection of vector fields that are nor-mal to M. Consequently, the Gauss and Weingarten formulas can be articulated as follows:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.7)$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X Y, \quad (2.8)$$

for any vectors $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ denotes the connection in the normal bundle. The second fundamental form h and the shape operator A_N are interconnected through the following equation:

$$g(A_N X, Y) = g(h(X, Y), N). \quad (2.9)$$

For any vectors $X, Y \in TM$ and $N \in T^{\perp}M$, we denote the following relations:

$$\varphi X = TX + NX, \quad (TX \in TM, NX \in T^{\perp}M), \quad (2.10)$$

$$\varphi N = tN + nN, \quad (tN \in TM, nN \in T^{\perp}M) \quad (2.11)$$

The submanifold M is classified as invariant if the vector N is identically null; conversely, M is deemed anti-invariant if the operator T is identically null. From equations (2.3) and (2.10), we deduce the following relationship:

$$g(X, TY) = -g(TX, Y). \forall X, Y \in TM \quad (2.12)$$

The covariant derivatives of the tensor fields T and N are articulated as follows:

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (2.13)$$

$$(\nabla^{\perp}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \quad (2.14)$$

for all vectors $X, Y \in TM$. The canonical structures T and N on a submani-fold M are considered to be *parallel* if $\nabla T = 0$ and $\nabla N = 0$, respectively. We consistently assume that the vector ξ is tangential to M. The submanifold M is classified as *invariant* if N is identically null, which implies that $\phi X \in TM$ for any vector $X \in TM$. Conversely, M is classified as *anti-invariant* if T is identi-cally null, thus $\phi X \in T^{\perp}M$ for any vector $X \in TM$. For each non-zero vector X that is tangential to M and is not proportional to ξ, we define $\theta(X)$ as the angle between ϕ and TX. The submanifold M is termed *slant* [4] if the angle $\theta(x)$ remains constant for all vectors $X \in TM - \{\xi\}$. The angle θ is referred to as the *slant angle* or *Wiritinger angle*. It is evident that if $\theta = 0$, then M is invariant, whereas if $\theta = \pi/2$, then M constitutes an anti-invariant submanifold. If the slant angle of M deviates from both 0 and $\pi/2$, it is designated as *proper slant*. A characterization of slant submanifolds is presented in the following theorem:

Theorem 2.1: [4] *Let M be a submanifold of an almost contact metric manifold M, such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \quad (2.15)$$

Moreover, if θ represents the slant angle, then it holds that $\lambda = \cos^2\theta$.

The subsequent relationships are direct implications of equation (2.15):

$$g(TX, TY) = \cos^2\theta\,[g(X, Y) - \eta(X)\eta(Y)], \quad (2.16)$$

$$g(NX, NY) = \sin^2\theta\,[g(X, Y) - \eta(X)\eta(Y)], \quad (2.17)$$

for arbitrary $X, Y \in TM$. We denote that M constitutes a semi-slant submani-fold of $\widetilde{M}$ if there exists an orthogonal direct decomposition of TM expressed as $TM = D_1 \oplus D_2$

$\oplus \{\xi\}$, wherein D_1 represents an invariant distribution, that is, $\phi(D_1) = D_1$, while D_2 is characterized as slant with a slant angle $\theta = 0$. The orthogonal complement of ND_2 corresponds to the normal bundle $T^{\perp}M$, which
is recognized as an invariant subbundle of $T^{\perp}M$ and is denoted by μ. Conse-quently, we derive that $T^{\perp}M = ND_2 \oplus \mu$. In a similar vein, we assert that M
qualifies as an anti-slant submanifold of $\tilde{M}$ if D_1 functions as an anti-invariant distribution of M, specifically, if $\phi D_1 \subseteq T^{\perp}M$, while D_2 remains slant with a slant angle $\theta \neq 0$.

3. Warped and doubly product manifolds

Let (N_1, g_1) and (N_2, g_2) denote two Riemannian manifolds, and let f be a positive differentiable function defined on N_1. The warped product of N_1 and N_2 is designated as the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \tag{3.1}$$

A warped product manifold $N_1 \times_f N_2$ is classified as trivial if the warping function f is constant. We recall the subsequent general formulation concerning warped products (refer to) [5].

$$\nabla_X V = \nabla_V X = (X \ln f) V, \tag{3.2}$$

where X is tangent to N_1 and V is tangent to N_2.
Let $M = N_1 \times_f N_2$ represent a warped product manifold, implying that N_1 is a totally geodesic submanifold and N_2 is a totally umbilical submanifold of M, respectively. Doubly warped product manifolds were conceived as an extension of warped product manifolds by B. Unal [14]. A doubly warped product manifold of N_1 and N_2, denoted as ${}_f2\ N_1 \times {}_f1\ N_2$, is endowed with a metric g defined as

$$g = f^2 g_1 + f^2 g_2. \tag{3.3}$$

Here, f_1 and f_2 are positive differentiable functions defined on N_1 and N_2, respectively. In this context, equation (3.2) is generalized as

$$\nabla_X V = (X \ln f_1) V + (V \ln f_2) X, \tag{3.4}$$

for each $X \in TN_1$ and $V \in TN_2$ (refer to) [15]. In the event that neither f_1 nor f_2 is constant, we encounter a non-trivial doubly warped product manifesting as $M = N_1 \times_f N_2$. It is evident that in such a scenario, both N_1 and N_2 are totally umbilical submanifolds of M.
For any vector fields $X \in TN_1$ and $Z \in TN_2$, by virtue of the equation (3.4), it follows that

$$\nabla_X Z = \nabla_Z X = (X \ln f_1) Z + (Z \ln f_2) X.$$

Assuming $\xi \in TN_1$, the aforementioned equation yields

$$\nabla_\xi Z = \nabla_Z \xi = (\xi \ln f_1) Z + (Z 1n f_2) \xi. \tag{3.5}$$

Conversely, employing equation (2.6) alongside the premise that ξ is tangent to
N_1, we derive

$$\tilde{\nabla}_{Z\xi} = \beta Z.$$

Utilizing equation (2.7), we ascertain

$$\nabla_Z \xi + h(Z, \xi) = \beta Z.$$

By employing equation (3.5) and subsequently comparing the tangential and normal components, we attain

$$(\xi \ln f_1) Z + (Z \ln f_2) \xi = \beta Z, \tag{3.6}$$

$$h(Z, \xi) = 0. \tag{3.7}$$

By taking the product with Z in equation (3.6) and utilizing the fact that the vector fields ξ, Z, and TZ are mutually orthogonal, it follows that $(\xi \ln f_1) = 1$ and $(Z \ln f_2) = 0$. This demonstrates that the function f_2 is indeed constant. Similarly, if the structural vector field ξ is tangent to N_2, for any $X \in TN_1$, we derive $(\xi \ln f_2) = 1$ and $(X \ln f_1) = 0$, thereby indicating that the function f_1 is constant. This leads us to the following theorem:

Theorem 3.1: *There do not exist proper doubly warped product submanifolds $M = {}_f2\ N_1 \times {}_f1\ N_2$ within a Kenmotsu manifold $\tilde{M}$, where N_1 and N_2 are any Riemannian submanifolds of the manifold $\tilde{M}$.*
The subsequent corollary is a direct implication of the preceding theorem:

Corollary 3.2: *There do not exist warped product submanifolds $M = N_1 \times_f N_2$ within a Kenmotsu manifold $\tilde{M}$ such that $\xi \in TN_2$, where N_1 and N_2 are any Riemannian submanifolds of $\tilde{M}$.*
Consequently, the only remaining case for examination pertains to warped prod-uct submanifolds $M = N_1 \times_f N_2$ of a Kenmotsu manifold $\tilde{M}$, such that $\xi \in TN_1$. For any $X \in TN_1$ and $Z \in TN_2$, we have

$$\left(\tilde{\nabla}_X \phi\right) = \nabla_X \phi Z - \phi \nabla_X Z.$$

Employing equation (2.6) and the premise that $\xi \in TN_1$, the left-hand side of the aforementioned equation is nullified due to the orthogonality inherent between the two distributions; consequently, it follows that $\phi \tilde{\nabla}_X Z = \tilde{\nabla}_X \phi Z$. Through the application of equations (2.7), (2.8), (2.10), and (2.11), we derive the expression

$$\nabla_X TZ + h(X, TZ) - A_{NZ} X + \nabla^{\perp}_X NZ = T \nabla_X Z + N \nabla_X Z + th(X, Z) + nh(X, Z).$$

By equating the tangential and normal components and utilizing (3.2), we obtain

$$A_{NZ} X = -th(X, Z) \tag{3.8}$$

And

$$\nabla^{\perp}_X NZ = (X \ln f) NZ + nh(X, Z) - h(X, TZ). \tag{3.9}$$

This progression culminates in the subsequent theorem:

Theorem 3.3: *Let $M = N_1 \times_f N_2$ represent warped product submanifolds of a Kenmotsu manifold $\tilde{M}$, under the condition that $\xi \in TN_1$, where N_1 and N_2 denote arbitrary Riemannian submanifolds of $\tilde{M}$. Then*

$$\xi \ln f = 1, \tag{3.10}$$

$$A_{NZ}X = -Bh(X, Z), \tag{3.11}$$

$$g(h(X, Y), NZ) = g(h(X, Z), NY), \tag{3.12}$$

$$g(h(X, Z), NW) = g(h(X, W), NZ), \tag{3.13}$$

for any $X, Y \in TN_1$ and $Z, W \in TN_2$.

4. Warped product semi-slant submanifolds

It has been established that the warped products of the form $N_1 \times_f N_2$ associated with Kenmotsu manifolds are non-existent in cases where $\xi \in TN_2$. Thus, in the subsequent section, we shall investigate warped product semi-slant submanifolds $N_1 \times_f N_2$ of Kenmotsu manifolds exclusively under the condition that $\xi \in TN_1$. If the manifolds N_θ and N_T (respectively $N_\perp$) are categorized as slant and invariant (respectively anti-invariant) submanifolds of the Kenmotsu manifold $\tilde{M}$, then their warped product semi-slant submanifolds can be expressed in one of the following configurations:

$N_T \times_f N_\theta$, $N_\perp \times_f N_\theta$, $N_\theta \times_f N_T$ and $N_\theta \times_f N_\perp$.

Let us consider the warped products of the type $N_T \times_f N_\theta$; subsequently, for any vector fields $X \in TN_T$ and $Z \in TN_\theta$, the following relationship holds true:

$$g(\phi\tilde{\nabla}_X Z, \phi Z) = g(\tilde{\nabla}_X Z, Z).$$

By employing equation (3.2), we derive

$$g(\phi\tilde{\nabla}_X Z, \phi Z) = (X \ln f)\|Z\|^2. \tag{4.1}$$

Conversely, we have

$$(\nabla^{\sim}_X \phi)Z = \nabla^{\sim}_X \phi Z - \phi\nabla^{\sim}_X Z,$$

applicable for any $X \in TN_T$ and $Z \in TN_\theta$. Utilizing equation (2.6) and recognizing that $\xi \in TN_T$, the left-hand side of the aforementioned equation equates to zero, thereby implying that $\tilde{\nabla}_X \phi Z = \phi\tilde{\nabla}_X Z$. By taking the inner product with ϕZ and subsequently employing equation (2.10), we arrive at

$$g(\phi\tilde{\nabla}_X Z, \phi Z) = g(\nabla^{\sim}_X (TZ + NZ), TZ + NZ)$$

By applying (2.9), (3.2), along with the Gauss and Weingarten formulas, we obtain

$$g(\phi\tilde{\nabla}_X Z, \phi Z) = (X \ln f)g(TZ, TZ) + g((\tilde{\nabla}_X N)Z, NZ) + (X \ln f)g(NZ, NZ).$$

Utilizing equations (2.16) and (2.17), we deduce

$$g(\phi\tilde{\nabla}_X Z, \phi Z) = (X \ln f)\|Z\|^2 + g((\tilde{\nabla}_X N)Z, NZ). \tag{4.2}$$

By substituting equations (4.1) and (4.2), we derive

$$g((\tilde{\nabla}_X N)Z, NZ) = 0. \tag{4.3}$$

Given that N_θ constitutes a proper slant submanifold of $\tilde{M}$, the resultant impli-cations of equation (4.3) lead to the following theorem:

Theorem 4.1: *Consider the manifold $M = N_T \times_f N_\theta$, which is characterized as a warped product semi-slant submanifold of the Kenmotsu manifold $\tilde{M}$ with the stipulation that $\xi \in TN_T$. Consequently, it follows that $(\tilde{\nabla}_X N)Z$ resides within the invariant normal subbundle for all vectors $X \in TN_T$ and $Z \in TN_\theta$, where the submanifolds N_T and N_θ are recognized as invariant and proper slant submanifolds of $\tilde{M}$.*

Let us consider the warped products of the form $N_\perp \times_f N_\theta$, where by for any vector $Z \in TN_\perp$ and $X \in TN_\theta$, we derive the following relation:

$$(\tilde{\nabla}_{X\phi})Z = \nabla^{\sim}_X \phi Z - \phi\nabla^{\sim}_X Z..$$

By employing equations (2.7), (2.8), (2.10), (2.11), (2.6), alongside the premise that $\xi \in TN_\perp$, we arrive at the following expression:

$$-\eta(Z)TX - \eta(Z)NX = -A_{NZ}X + \nabla^{\perp}_X NZ - T\nabla_X Z - N\nabla_X Z - th(X, Z) - nh(X, Z).$$

Utilizing the tangential components in conjunction with equation (3.2), we ob-tain:

$$\eta(Z)TX = A_{NZ}X + (Z \ln f)TX + th(X, Z).$$

By taking the inner product with TX in the preceding equation and recognizing that the vector fields X and TX are orthogonal to one another, it follows that:

$$\eta(Z)g(TX, TX) = g(A_{NZ}X, TX) + (Z \ln f)g(TX, TX) + g(th(X, Z), TX).$$

Therefore, from equations (2.9) and (2.16), we derive the following expression:

$$\{\eta(Z) - (Z \ln f)\}\cos^2\theta\|X\|2 = g(h(X, TX), NZ) - g(h(X, Z), NTX). \tag{4.4}$$

As N_θ is established as a proper slant submanifold, by interchanging X with TX in the aforementioned equation and considering equation (2.15), we deduce the following result:

$$\{\eta(Z) - (Z \ln f)\}\cos^2\theta\|X\|^2 = -g(h(X, TX), NZ) + g(h(TX, Z), NX). \tag{4.5}$$

Upon the summation of equations (4.4) and (4.5), one derives

$$\{\eta(Z) - (Z \ln f)\}\cos^2\theta\|X\|2 = -g(h(X, TX), NZ) + g(h(TX, Z), NX).$$

Consequently, by virtue of (3.13), the right-hand side of the aforementioned
equation evaluates to zero, yielding

$\{\eta(Z) - (Z \ln f)\} \cos^2 \theta \|X\| 2 = 0.$

This paves the way for the subsequent theorem:

Theorem 4.2: *Consider $M = N_\perp \times_f N_\theta$ as warped product semi-slant submani-folds of the Kenmotsu manifold $\tilde{M}$ under the condition that $\xi \in TN_\perp$. Therefore,*

$$\eta(Z) = (Z \ln f) . \quad (4.6)$$

This assertion holds for all $Z \in TN_\perp$ where $N_\perp$ and N_θ are characterized as anti-invariant and proper slant submanifolds of M, respectively.
Let $M = N_\theta \times_f N_T$ be defined as warped product semi-slant submanifolds of a Kenmotsu manifold M with the condition that $\xi \in TN_\theta$. Consequently, for any vector $X \in TN_T$ and $Z \in TN_\theta$, it follows that

$$(\tilde{\nabla}_X \phi)Z = \tilde{\nabla}_X \phi Z - \phi \tilde{\nabla}_X Z$$

Utilizing equations (2.7), (2.8), (2.10), (2.11), (2.6) and acknowledging the condition $\xi \in TN_\perp$, we deduce

$$-\eta(Z)\phi X - \eta(Z)\phi X = \nabla_X TZ + h(X, TZ) - A_{NZ}X + \nabla^\perp_X NZ - T\nabla_X Z - N\nabla_X Z - th(X, Z) - nh(X, Z) .$$

By contrasting the tangential and normal components, we arrive at

$$\eta(Z)\phi X = -A_{NZ}X + \nabla_X TZ - T\nabla_X Z - th(X, Z) \quad (4.7)$$

and

$$(\tilde{\nabla}_X N)Z = nh(X, Z) - h(X, TZ) . \quad (4.8)$$

Taking the inner product with NZ in (4.8), we derive the following expression:
$g((\tilde{\nabla}_X N)Z, NZ) = g(nh(X, Z), NZ) - g(h(X, TZ), NZ) .$
$= g(\phi h(X, Z), \phi Z) - g(th(X, Z), TZ) - g(h(X, TZ), \phi Z) .$
Consequently, we arrive at the following conclusion:

$$g((\tilde{\nabla}_X N)Z, NZ) = -g(th(X, Z), TZ) + g(th(X, TZ), Z) . \quad (4.9)$$

Given that $\theta \neq \frac{\pi}{2}$, we substitute Z with TZ in (4.9) and apply (2.15), resulting in the expression:

$$g((\tilde{\nabla}_X N)TZ, NTZ) = \cos^2\theta \{-g(th(X, Z), TZ) + g(th(X, TZ), Z)\} .$$
By utilizing equation (4.9), we obtain:

$$g((\tilde{\nabla}_X N)TZ, NTZ) = \cos^2\theta\, g((\tilde{\nabla}_X N)Z, NZ) .$$

Therefore, we can articulate the subsequent theorem:

Theorem 4.3: *Let $M = N_\theta \times_f N_T$ represent warped product semi-slant subman-ifolds of the Kenmotsu manifold $\tilde{M}$ such that $\xi \in TN_\theta$. Then*

$$g((\tilde{\nabla}_X N)Z, NZ) = \sec^2\theta\, g((\tilde{\nabla}_X N)TZ, NTZ) . \quad (4.10)$$

for all $X \in TN_T$ and $Z \in TN_\theta$ where N_T and N_θ are invariant and proper slant submanifolds of $\tilde{M}$, respectively.
Let $M = N_T \times_f N_\perp$ denote warped product semi-slant submanifolds of the Kenmotsu manifold $\tilde{M}$ such that $\xi \in TN_\theta$. Thus, for any $X \in TN_T$ and $Z \in TN_\perp$, we have:

$$(\tilde{\nabla}_X \phi)Z = \nabla_X \phi Z - \phi\nabla_X Z.$$

By applying equation (2.6) and acknowledging that $\xi \in TN_\perp$, we deduce:

$$\nabla_X \phi Z = \phi\nabla_X Z.$$

Utilizing equations (2.7), (2.8), (2.10), and (2.11), we arrive at the expression:

$$-A_{NZ}X + \nabla^\perp_X NZ = T\nabla_X Z + N\nabla_X Z + th(X, Z) + nh(X, Z) .$$

From the normal components of the aforementioned equation and applying for-mula (3.2) yields:

$$\nabla^\perp_X NZ = (X \ln f)NZ + nh(X, Z) . \quad (4.11)$$

Taking the product in (4.11) with NW_1 for any$W_1 \in TN_\perp$, we derive the follow-ing expression:

$$g(\nabla^\perp_X NZ, NW_1) = (X \ln f)\, g(NZ, NW_1) + g(nh(X, Z), NW_1) .$$
Alternatively, we may express this as:

$$g(\nabla^\perp_X NZ, NW_1) = (X \ln f)\, g(\phi Z, \phi W_1) + g(\phi h(X, Z), \phi W_1) .$$

From equation (2.2), we can assert that

$$g(\nabla^\perp_X NZ, NW_1) = (X \ln f)\, g(Z, W_1) . \quad (4.12)$$

Conversely, for any $X \in TN_T$ and $Z \in TN_\perp$, we find the relationship:

$$(\tilde{\nabla}_X N), Z = \nabla^\perp_X NZ - N\nabla_X Z.$$

By taking the product with NW_1, for any $W_1 \in TN_\perp$ and employing equation (3.2), we obtain

$$g((\tilde{\nabla}_X NZ), NW_1) = g(\nabla^\perp_X NZ, NW_1) - (X \ln f)\, g(Z, W_1) \quad (4.13)$$

The implications of equations (4.12) and (4.13) yield the conclusion that

$$g((\tilde{\nabla}_X N)Z, NW_1) = 0, \quad (4.14)$$

for any $X \in TN$ and Z, $W_1 \in TN^{\perp}$. In the event that $W_2 \in TN_T$, utilizing the formula (2.14), one can deduce that

$g((\nabla^{\tilde{}}X\,N)\,Z, \phi W_2) = g(\nabla^{\perp}_X NZ, \phi W_2) - (X \ln f)\, g\,(Z, \phi W_2)$.

Given that N_T represents an invariant submanifold, it follows that $\phi W_2 \in TN_T$ for any $W_2 \in TN_T$. Thus, taking into account the property that the scalar product of the tangential component with the normal component is null, we arrive at the following result:

$$g((\bar{\nabla}_X N)Z, NW_1) = 0, \qquad (4.15)$$

for any X, $W_2 \in TN_T$ and $Z \in TN^{\perp}$. Consequently, the equations (4.14) and (4.15) culminate in the following theorem:

Theorem 4.4: *Let $M = N_T \times_f N^{\perp}$ be warped product semi-slant submanifolds of the Kenmotsu manifold $\bar{M}$ such that $\xi \in TN_T$. Then it follows that $(\bar{\nabla}_X N)$ Z resides within the invariant normal subbundle for all $X \in TN_T$ and $Z \in TN^{\perp}$, wherein N_T and $N^{\perp}$ are characterized as invariant and anti-invariant subman-ifolds of $\bar{M}$, respectively.*

5. References

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