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Solving singularity structures in nonlinear ordinary and partial differential equations

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Abstract
The study explores various mathematical tools, such as transform methods, integral equations, and regularization techniques, to characterize and resolve singular behavior in PDEs. Emphasis is placed on developing robust numerical algorithms capable of accurately capturing and resolving singularities in complex PDE systems. Furthermore, the thesis addresses the application of singularity analysis in interdisciplinary domains including nonlinear optics, fluid dynamics, and mathematical biology. Case studies illustrate the practical relevance of the proposed methodologies in modeling physical phenomena characterized by singular behavior, shedding light on the underlying mechanisms and offering insights for engineering design and scientific inquiry. Overall, this thesis contributes to advancing the understanding and treatment of singularity structures in nonlinear ODEs and PDEs, offering valuable insights into the nature of singularities and providing effective computational tools for tackling challenging problems across various domains of science and engineering.

Keywords: Generalized boussinesq and kaup-kupershmidt equations, Cheng equation, conservation laws

Introductions

Our short discussion how can construct similarity solutions with ease by using the theory of differential invariants to the notion of Lie symmetries of differential equations.
Consider Φ as the map of a point transformation with one parameter, like

$$\Phi\left(u^{A}(t,x)\right)=u'^{A}(t,x) \tag{13}$$

being an infinitesimal transformation, where ε is the smallness parameter.

$$t'=t+\varepsilon\xi^{1}\left(t,x,u^{B}\right) \tag{14}$$

$$x'=x+\varepsilon\xi^{2}\left(t,x,u^{B}\right) \tag{15}$$

$$y'=u^{A}+\varepsilon\eta^{A}\left(t,x,u^{B}\right) \tag{16}$$

and generator

$$X=\frac{\partial t'}{\partial\varepsilon}\partial_{t}+\frac{\partial x'}{\partial\varepsilon}\partial_{x}+\frac{\partial u^{A'}}{\partial\varepsilon}\partial_{u^{A}} \tag{17}$$

in which $u^{A}(t,x)=(v(t,x),w(t,x))$.

Consider now that $u^{A}(t,x)$ offers a response to the PDE

$$\mathscr{H}\left(u^{A},u^{A}_{,t},u^{A}_{,x}\ldots\right)=0.$$

Then, operating inside the Φ-defined map $u^{A'}(t', x')$ as well as being a differential problem solution

$$\mathscr{H}\left(u^A, u^A_{,t}, u^A_{,x} \ldots\right) = 0\,\mathrm{i}$$

Provided that.

$$\Phi\left(\mathscr{H}\left(u^A, u^A_{,t}, u^A_{,x} \ldots\right)\right) = 0,$$

Therefore, the equation involving differences.

$$\mathscr{H}\left(u^A, u^A_{,t}, u^A_{,x} \ldots\right) = 0$$

Remains constant regardless of how the map, Φ, is applied. If this is correct, the differential equations Lie symmetry holds for the generator, X, of the infinitesimal transformation of the one-parameter point transformation, Φ.

$$\mathscr{H}\left(\bar{u}^A, u^A_{,t}, u^A_{,x} \ldots\right) = 0.$$

$$X^{[n]}(\mathscr{H}) = 0, \tag{18}$$

or equivalently

$$X^{[n]}(\mathscr{H}) = \psi\mathscr{H}, \mathrm{mod}(\mathscr{H}) = 0, \tag{19}$$

Generalized Boussinesq and Kaup-Kupershmidt Equations

Take into account the Boussinesq equation.

$$u_{tt} - u_{xx} + (u^2)_{xx} \pm u_{4x} = 0,$$

When waves in shallow water are shown. This equation's most prominent behaviour is.

$$u(x,t) \sim \quad \mp 6g_x^2(x,t)g^{-2}(x,t)$$

With resonances at.

$$r_1 = -1, r_2 = 4, r_3 = 5, \text{ and } r_4 = 6.$$

A general solution to its series is.

$$\begin{aligned} u(x,t) = &\mp \frac{6g_x^2(x,t)}{g^2(x,t)} \pm \frac{6g_{xx}(x,t)}{g(x,t)} - \Big(g_t^2(x,t) - g_x^2(x,t) - 3g_{xx}^2(x,t) \\ &+ 4g_x(x,t)g_{3x}(x,t)\Big)\frac{1}{2g_x^2(x,t)} + \Big(g_{tt}(x,t)g_x^2(x,t) - g_t^2(x,t)g_{xx}(x,t) \\ &+ 3g_{xx}^3(x,t) - 4g_x(x,t)g_{xx}(x,t)g_{3x}(x,t) + g_x^2(x,t)g_{4x}(x,t)\Big)\frac{g(x,t)}{2g_x^4(x,t)} \\ &+ u_4(x,t)g^2(x,t) + u_5(x,t)g^3(x,t) + u_6(x,t)g^4(x,t) + \cdots, \end{aligned}$$

Were,

$$g(x,t), u_4(x,t), u_5(x,t) \text{ and } u_6(x,t)$$

may be defined in any way. Therefore, the Painlev'e test is satisfied by equation.

Cheng Equation

For the area of photosensitive molecules, Cheng addressed two nonlinear partial differential equations in 1984, namely.

$$\begin{aligned} u_x &= -auv \\ v_t &= bu_x, \end{aligned}$$

$u(t,x)$ and $v(t,x)$

Where symbolize the light's intensity and the molecules' density, correspondingly. Specifically, the first equation shows that when light is absorbed, the density of molecules increases with time, Light intensity multiplied by molecular density is precisely proportional to the amount of light absorbed by molecules in a film when exposed to a light beam, according to the second equation.

Absorption and proportionality are both described by the constants a and b, respectively. The second system is significant because it provides the first example of precisely solitons solving a failing film, where the waves differ from the solitons, which were experimentally seen by Kapitza & Kapitza in 1949. We direct the reader to a review of the solitary waves that includes some further research on the topic. As a comprehensive analysis of solo waves on underperforming films is presented. Previous research used Hirota's bilinearization approach and to solve the Cheng issue using Painlev'e analysis. Also discussed were the generic travelling wave solution of (1.1). Using Lie's symmetry analysis approach, we review this problem in this work and talk about potential system reductions and how to solve them. We also go over the basic Lie algebraic structure.

The Cheng Equation may be simplified to the Riccati Equation, which is a linear nonhomogeneous equation of the form of the one proposed by Euler, and to the First and Second forms About Abel's equations, which may be resolved using special functions as their solutions. As far as the authors are aware, there has been no prior discussion in the literature reducing the Cheng issue to linear equations of the kind Abel and Euler. Therefore, this is our primary finding. It bears repeating that we discovered a novel general solution type that is not the end product of the broad approaches discussed in the literature. A more comprehensive analysis is produced using Lie's method. The situation in which the additionally, the research delves into the space-dependent parameters of the Cheng Equation. Here is the structure of the paper. Chapter 2 presents the Lie symmetries of equation (1.1).

In the sections that follow, we cover the space-dependent parameter case as well as the symmetry analysis that corresponds to different examples of arbitrary functions in point symmetries. Subsequently, the Conclusion and appropriate references are cited.

Cheng equation symmetry calculation

A short description of the Lie point symmetries theory is provided for the reader's convenience. Specifically, we outline the fundamental concepts and primary Method for determining the Lie point symmetries of a certain differential equation.

Think about

$$H^A\left(t,x,u^A,u^A_{,i}\right)=0,$$

As a group of differential equations, where.

(u,v) and $u^A_{,i}=\frac{\partial u^A}{\partial y^i}$ in which $y^i=(t,x)$.

$u^A =$

$$\begin{aligned} t' &= t\left(t,x,u^A;\varepsilon\right) \\ x' &= x\left(t,x,u^A;\varepsilon\right) \\ u'^A &= u^A\left(t,x,u^A;\varepsilon\right) \end{aligned}$$

Next, the little one-parameter point transformation comes into play.

ε

In which constitutes a negligible variable, and the system of differential equation H^A is invariant.

$$H^A\left(t', x', u'^A\right) = H^A\left(t, x, u^A\right)$$

Or equivalent

$$\lim_{\varepsilon \to 0} \frac{H^A\left(t', x', u'^A; \varepsilon\right) - H^A\left(t, x, u^A\right)}{\varepsilon} = 0.$$

The Lie derivative is defined by the latter equation $\mathcal{L}$ of H^A.
In the second statement, the Lie derivative is defined in terms of the direction.

$$\Gamma = \frac{\partial t'}{\partial \varepsilon}\partial_t + \frac{\partial x'}{\partial \varepsilon}\partial_x + \frac{\partial u^A}{\partial \varepsilon}\partial_{u^A}.$$

Therefore, we may state that the set of differential equations will have a Lie point symmetry in the vector field H^A Γ In some circumstances.

$$\mathcal{L}_\Gamma\left(H^A\right) = 0.$$

For each function, a set of differential equations that are linear in nature.

$$\xi^t = \frac{\partial t'}{\partial \varepsilon}, \; \xi^x = \frac{\partial x'}{\partial \varepsilon} \text{ and } \eta^A \frac{\partial u^A}{\partial \varepsilon},$$

whose answers form the Truth point symmetries explicitly. The computations are skipped and we state that the Lie symmetries are provided by subjecting the system to the Lie symmetry condition (2.7) (1.1).

$$\begin{aligned}\Gamma_1 &= -vg'(x)\partial_v + g(x)\partial_x \\ \Gamma_2 &= h(t)\partial_t - uh'(t)\partial_u,\end{aligned}$$

$2A_1$ ^{1}. $h(t)$ and $g(x)$ Where may be defined in any way. The system's accepted Lie algebra is.
We do the reductions after considering several scenarios involving the arbitrary functions.
$h(t)$ and $g(x)$ a. consist of constant functions that provide the symmetries for translation.
$h(t)$ and $g(x)$ b. provide the scaling symmetries and are hence the identical function.
$h(t)$ AND $g(x)$ c. In a broad sense

3. Case I: are constant functions

$$\begin{aligned}\Gamma_{1A} &= \partial_x \\ \Gamma_{2A} &= \partial_t.\end{aligned}$$

$c\Gamma_{1A} + \Gamma_{2A}$ Provides an answer based on travelling waves. Variables that measure similarity include

$$\begin{aligned}f &= x - ct \\ u(x,t) &= w(f) \\ v(x,t) &= k(f),\end{aligned}$$

$$f$$

Where represents a new independent variable, and.

$$w(f) \text{ and } k(f)$$

May be relied upon from this point on. As a result, the system's output

$$\begin{aligned} w'(f) &= -ak(f)w(f) \\ ck'(f) &= -bw'(f). \end{aligned}$$

One way to express system (3.3) considering it as a second-order equation in relation to w, t in particular, to.

$$\frac{-cw'(f)^2}{aw(f)^2} + \frac{cw''(f)}{aw(f)} = bw'(f).$$

$$\begin{aligned} \Gamma_{1B} &= \partial_f \\ \Gamma_{2B} &= f\partial_f - w\partial_w. \end{aligned}$$

The symmetries of (3.4) are.
We consider Γ_{1B} in order to minimize. The original, standard coordinates are.

$$n = w(f) \quad \text{and} \quad m(n) = \frac{1}{w'(f)}.$$

The result is a decrease in (3.4) to.

$$m'(n) = -\frac{nm(n)^2ab}{c} - \frac{m(n)}{n}.$$

The invariants that vary,

$$n = w(f) \quad \text{and} \quad m = w'(f),$$

Cut (3.4) in half to.

$$m'(n) = \frac{nab}{c} + \frac{m(n)}{n}.$$

Both equations (3.7) and (3.6) are linear equations of the type Euler, on the other hand, the Riccati equation is (3.7). It is then taken into consideration Γ_{2B} in order to minimize. The original, standard coordinates are.

$$n = fw(f) \quad \text{and} \quad m(n) = \frac{1}{f(fw'(f) + w(f))}.$$

A reduction of (3.4) to.

$$m'(n) = \frac{(n^3ab + cn^2)m(n)^3}{cn} + \frac{(-n^2ab - cn)m(n)^2}{cn} - \frac{m(n)}{n}.$$

The differential invariants, reduce to.

$$n = fw(f) \quad \text{and} \quad m(n) = f^2w'(f)),$$

$$m'(n) = \frac{m(n)(n^2ab + m(n) + 2cn)}{cn(m(n) + n)}.$$

Examining the Kaup-Kupershmidt (KK) equation brings us to this point.

$$u_t = 5u^2u_x + \frac{25}{2}u_xu_{2x} + 5uu_{3x} + u_{5x}.$$

All terms must be of the same rank for the dilation symmetry to be found in equation.

$$w(u) + w(D_t) = 3w(u) + w(D_x) = 2w(u) + 3w(D_x)$$
$$= 2w(u) + 3w(D_x) = w(u) + 5w(D_x).$$

If we set $w(D_x) = 1$, then $w(u) = 2, w(D_t) = 5$

And that ranks as 7. The first six generalised symmetries are ranked using these weights as follows.

$$\text{rank } G^{(1)} = 3, \quad \text{rank } G^{(2)} = 7, \quad \text{rank } G^{(3)} = 9,$$
$$\text{rank } G^{(4)} = 13, \quad \text{rank } G^{(5)} = 15, \quad \text{rank } G^{(6)} = 19.$$

We guess that $\mathcal{R} = 6$ and $s = 2$, since rank $G^{(2)}$ − rank $G^{(1)} \neq$ rank $G^{(3)}$ − rank.

rank $G^{(2)}$ but rank $G^{(3)}$ − rank $G^{(1)}$ = rank $G^{(4)}$ − rank $G^{(2)} = 6$.

Thus, taking all $D_x^i u^j$ $(i, j \in \mathbb{Z}^*)$ such that rank $D_x^i u^j = 6$ gives

$$\mathcal{R}_0 = c_1D_x^6 + c_2uD_x^4 + c_3u_xD_x^3 + (c_4u^2 + c_5u_{2x})D_x^2$$
$$+ (c_6uu_x + c_7u_{3x})D_x + (c_8u^3 + c_9u_x^2 + c_{10}uu_{2x} + c_{11}u_{4x})I.$$

The densities are computed using Invariants Symmetries. M.

$$\rho^{(1)} = u, \rho^{(2)} = 3u_x^2 - 4u^3$$

And the symmetries that are generalized.

$$G^{(1)} = u_x, G^{(2)} = F(u) = 5u^2u_x + \tfrac{25}{2}u_xu_{2x} + 5uu_{3x} + u_{5x}$$

We determine using these densities and generalized symmetries.

$$G^{(1)}D_x^{-1}\mathcal{L}_u(\rho^{(2)}) = u_xD_x^{-1}\mathcal{L}_u(3u_x^2 - 4u^3) = u_xD_x^{-1}(-6u_{2x} - 12u^2)$$

and

$$G^{(2)}D_x^{-1}\mathcal{L}_u(\rho^{(1)}) = F(u)D_x^{-1}\mathcal{L}_u(u) = \left(5u^2u_x + \frac{25}{2}u_xu_{2x} + 5uu_{3x} + u_{5x}\right)D_x^{-1}.$$

As a result, the potential integral operator is.

$$\mathcal{R}_1 = c_{12}u_xD_x^{-1}(u_{2x} + 2u^2)I + c_{13}\left(5u^2u_x + \frac{25}{2}u_xu_{2x} + 5uu_{3x} + u_{5x}\right)D_x^{-1}.$$

$$\mathcal{R}_1 = c_{12}u_x D_x^{-1}(u_{2x} + 2u^2)I + c_{13}\left(5u^2u_x + \frac{25}{2}u_xu_{2x} + 5uu_{3x} + u_{5x}\right)D_x^{-1}.$$

Lie Symmetries

$$F(\mathbf{x}, u_\alpha, u_\beta, u_\delta, u_{\alpha\beta}, \ldots) = 0,$$

Where $\mathbf{x} = (x_\alpha, x_\beta, x_\delta)$. are the set of independent variables.

$$u_\alpha = \frac{\partial u}{\partial x_\alpha},\ u_\beta = \frac{\partial u}{\partial x_\beta},\ \text{and}\ u_{\alpha\beta} = \frac{\partial u}{\partial x_\alpha x_\beta},$$

And moving forward, the following words may be specified. Given a change of a single point on the limited scale.

$$\begin{aligned}
\bar{x_\alpha} &= x_\alpha + \epsilon\xi^\alpha(x_\alpha, x_\beta, x_\delta, u) + \mathcal{O}(\epsilon^2),\\
\bar{x_\beta} &= x_\beta + \epsilon\xi^\beta(x_\alpha, x_\beta, x_\delta, u) + \mathcal{O}(\epsilon^2),\\
\bar{x_\delta} &= x_\delta + \epsilon\xi^\delta(x_\alpha, x_\beta, x_\delta, u) + \mathcal{O}(\epsilon^2),\\
\bar{u} &= u + \epsilon\eta(x_\alpha, x_\beta, x_\delta, u) + \mathcal{O}(\epsilon^2),
\end{aligned}$$

In order for equation to be considered invariant, it must hold true.

$$F(\mathbf{x}, u) = F(\bar{\mathbf{x}}, \bar{u}).$$

Literature Review

Osman, *et al*. (2020) [11]. Important nonlinear physical models are examined in this study, the coupled Burgers equations (CBEs) in dimensions (2+1). Specifically, a set of the coupled Burgers equations in two waves and half dimensions are obtained by using the generalized unified method (GUM). A non-traveling wave solution and symmetry reductions of the coupled Burger's equations in (2+1) dimensions are also accomplished using the Lie symmetry method (LST). We looked at how the CBEs' double-wave solutions' wave structures changed for various parameter values using certain figures.

Shirai, Akira & Yoshino, Masafumi. (2008) [1]. Nonlinear partial differential equations are addressed by a theorem of the Frobenius type. A singular vector field's normal form theory is a common example of an application. There is a tight relationship between Riemann-Hilbert factorization and the development of singular solutions.

Paliathanasis, Andronikos & Leach, Pgl. (2016) [2]. The integrability of ordinary differential equations that are not linear equations is studied using two crucial methods: analysis of symmetry and singularities. Here we outline the key points where these two approaches are similar and distinct.

Ahmed, Muhammad & Naeem, Rishi & Tarar, Muhammad & Iqbal, Muhammad Sajid & Inc, Mustafa & Afzal, Farkhanda. (2023) [3]. Two important examples of stochastic functions are coefficient singularities and white noise obstacles in the present study of accurate applications of nonlinear PDEs are addressed. Modern scholarship has benefited from four significant additions. In mathematics, one uses Schauder's fixed point theorem to build clear estimates of the presence of solutions based on past information. The second aspect is the regulation of the solution's behaviour in the case when the singular parameter ϵ approaches zero. Thirdly, accurate solutions have effectively dealt with the influence of noise in the differential equation. Simulating the precise answers and explaining the graphs is the last contribution.

Hermann, Martin & Saravi, Mohsen. (2016) [5]. In this work, the authors provide analytical and numerical approximation approaches for finding solutions to ODEs that are nonlinear. Solving lower-order ordinary differential equations (ODEs), whether linear or nonlinear, has lately seen heavy usage of analytical approximation approaches. Also covered is the topic of solving powerful nonlinear ODEs using these approaches. Since systems of nonlinear ordinary differential equations with huge dimensions that are unsolvable analytical approximation techniques are prevalent in practice, two chapters are dedicated to numerical approaches for tackling these types of problems. Aside from that, it investigates analytical and numerical methods for solving ODEs that rely on parameters. A number of techniques for dealing with nonlinear oscillators and structural systems are detailed in the book. These techniques include the following: amplitude frequency formulation, energy balance, harmonic balance, techniques for nonlinear stabilised marching, simple and multiple shooting, homotopy analysis, iteration perturbation, variational iteration, and homotopy perturbation. Solving issues involving nonlinear oscillators and structural systems is the focus of this book, which delves deeply into a number of novel analytical and numerical approximation methods. After heavy reliance on the finite element approach, many students graduate from college ill-equipped to handle real-world situations. The book offers a number of novel approximation strategies to address this issue.

Bilinear Geometry and Wronskian Models

When the solutions are expressed in terms of a tau-function, the KP equation may be expressed in bilinear form. If $\tau(x, y, t)$

Then

$$\iota(x, y, t) = 2\partial_x^2 \log \tau(x, y, t),$$

Fulfils what the Hirota bilinear equation's state is.

D_x, D_y, D_t $D_x^m f \cdot g = (\partial_x - \partial_{x'})^m f(x, y, t) g(x', y, t)|_{x=x'},$ in which

$$\left[D_x D_t + D_x^4 + 3D_y^2\right] \tau \cdot \tau = 0,$$

Are derivatives of Hirota.

the like. In order to solve the KP problem, one may use Hirota's approach, which is based on this formulation [see Hirota (2004)]. The tau-function, when expressed in Wronskian form, also provides solutions to Hirota's bilinear problem.

$$\tau(x, y, t) = \mathrm{Wr}(f_1, \ldots, f_N) = \det \begin{pmatrix} f_1^{(0)} & \cdots & f_N^{(0)} \\ \vdots & \ddots & \vdots \\ f_1^{(N-1)} & \cdots & f_N^{(N-1)} \end{pmatrix},$$

$f^{(n)}$

$x,$ in which marks the halfway point between f

In reference to $f_1, \ldots, f_N$

As well as when the related linear equation is solved by the functions.

$$\sigma f_y + f_{xx} = 0, \qquad f_t + 4f_{xxx} = 0.$$

The Lax pair (2)-(3) is symbolised for in this system of linear equations.

$$\lambda = \alpha = 0 \text{ and } u \equiv 0.$$

This allows one to get a wide range of precise solutions to the KP equation. Below, we will quickly go over a few of them.

The conservation laws and similarity solutions for the beam equations a full experiment

Beams may be broadly classified into two sorts. There is one kind that works on both ends and another kind that works on only one. Here we have a cantilever. Since the latter may have its free end oscillate, it is more physically and mathematically interesting. As a result, the beam experiences strains. While Daniel Bernoulli and Leonhard Euler provided the first mathematical account in around 1750, previous efforts by Galileo Galilei and Leonardo da Vinci were somewhat hindered due to their lack of understanding of differential equations. Leonhard Euler and Daniel Bernoulli were both influenced by Jacob Bernoulli. Lord Rayleigh, a mathematician, suggested a modification to the Euler-Bernoulli model in 1894 that included a component pertaining to rotational stress. Timoshenko made significant advancements to the model that is now known as the Timoshenko-Prescott model in 1921.

Comparing theoretical predictions with actual findings has been the focus of a great deal of experimental and numerical research. Notably, all three of the aforementioned models are linear, contrary to the three-dimensional infinitesimal theory of elasticity. Although they simplify mathematics, they are not without cost. It is strange that some practitioners still choose the most basic model, the one put forward by Euler and Bernoulli.

In an effort to compare experimental results with theoretical predictions, several studies have attempted to simulate a one-dimensional model as closely as feasible. Examining how shock waves travel along a beam is an intriguing area of research. In order to simulate A small-diameter (25 mm) bullet is shot into the fixed end of the cantilever beam to provide a homogeneous boundary condition.

Lie symmetry analysis

The Euler-Bernoulli Equation

Being expressed in its forming the beam equation in the Euler-Bernoulli style.

$$\alpha\beta u_{xxxx} + u_{tt} = 0.$$

There are Lie point symmetries that are.

$$\begin{aligned}\Gamma_{1a} &= \partial_x \ , \ \Gamma_{2a} = \partial_t, \\ \Gamma_{3a} &= u\partial_u \ , \ \Gamma_{4a} = 2t\partial_t + x\partial_x, \\ \Gamma_{5a} &= a(t,x)\partial_u,\end{aligned}$$

In which case the beam equation in its Euler-Bernoulli form is satisfied by a (t, x). According to the Morozov-Mubarakzyanov classification system, "The Lie Algebra" (A3, 3 ⊕ A1) ⊕s ∐A1.

Conclusion

In this research, we studied and found analytical solutions to the Cheng problem using the theory of invariant transformations. Our research has shown that the Abel equations' solutions can be used to represent the similarity solutions. Specifically, our research led us to the conclusion that Abel's Equations are synonymous with the Cheng equation. This equation undergoes a change from second-order partial differential equation to first-order ordinary differential equation by the use of point transformations. It is also possible to represent the system solutions using the Lambert W functions. There is strong evidence that both explicit and implicit solutions exist due to the reductions to the various versions of Abel's equation.

Having conducted our study for a variety of similarity variable selections, we found that arbitrary functions are used by the symmetry vectors. In every case, the system can be reduced to either an Abel's or a Riccati Equation, hence the results are the same. The Lie algebra was also mentioned in the chapter. Lastly, additionally, we address the case when the parameters vary with respect to space. An approach based on Lie symmetry has provided a generic answer for the last equation. It is possible to examine the spatial and temporal dependency of the parameteric values in subsequent studies.

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